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AN EXACT SOLUTION OF THE COUPLED PHASE CHANGE PROBLEM IN A POROUS MEDIUM

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Abstract—An exact solution of the coupled heat and mass transfer in a porous medium with evaporation is given. The governing partial differential equations are transformed to ordinary differential equations by use of Boltzmann transformation of variables. Evaporation thickness is proportional to the square root of time, where the proportional constant λ is obtained from an equation which contains tabulated functions only. Effects of nondimensional parameters on evaporation speed are briefly discussed.

NOMENCLATURE

a_m, moisture diffusivity;
a_q, thermal diffusivity;
c_m, specific mass capacity;
c_q, specific heat capacity;
k, thermal conductivity;
Ko, Kossovitch number defined by equation (16);
L, latent heat of evaporation of liquid per unit mass;
Lu, Luikov number defined by equation (15);
$s(\tau)$, position of evaporation front;
t, temperature;
t_s, temperature at surface $x = 0$;
T, nondimensional temperature defined by equation (12);
x, space coordinate.

Greek symbols

ε, coefficient of internal evaporation;
η, dimensionless variable defined by equation (13);
θ, mass-transfer potential;
Θ, nondimensional mass-transfer potential defined by equation (11);
λ, dimensionless constant defined by equation (19);
ν, nondimensional latent heat of evaporation defined by equation (17);
ρ_m, density of moisture;
ρ_q, density of porous medium;
τ, time.

Subscripts

v, vaporizing state;
1, first region, $0 < x < s(\tau)$;
2, second region, $x < s(\tau)$;
21, ratio of properties of region 2 to region 1.

1. INTRODUCTION

HEAT-CONDUCTION problems involving melting, freezing, or evaporation have wide applications in foundry, welding, food technology, etc. Due to the inherent nonlinearity of the problem, few exact solutions are known. Similarity type exact solutions to the phase change problems exist when the surface temperatures are fixed [1–3]. Recently Gupta [4] presented an approximate solution to a coupled heat- and mass-transfer problem involving evaporation. His solution was obtained by the local potential method, and was compared to another approximate solution obtained by the integral technique [5]. The problem Gupta [4] treated has analytical solution, which is presented in this paper.

2. STATEMENTS OF THE PROBLEM AND THE SOLUTION

The problem studied here is the same as that of [4]. A semi-infinite porous medium is dried by maintaining the surface at a constant temperature t_s above the evaporation point. Initially, the whole body is at a uniform temperature t_0 and uniform moisture potential θ_0. The moisture is assumed to evaporate completely at a constant temperature, evaporation point t_v. It is also assumed that the moisture potential in the first region, $0 < x < s(\tau)$, is constant at θ_0, where $x = s(\tau)$ locates the evaporation front. It is further assumed that the moisture in vapor form does not take away any appreciable amount of heat from the system. Neglecting mass diffusion due to temperature variation, the problem can be expressed as

$$\frac{\partial t_1}{\partial \tau} = a_q \frac{\partial^2 t_1}{\partial x^2}, \quad 0 < x < s(\tau), \quad \tau > 0 \tag{1}$$

$$\theta_1 = \theta_v, \quad 0 < x < s(\tau), \quad \tau > 0 \tag{2}$$

$$\frac{\partial t_2}{\partial \tau} = a_q \frac{\partial^2 t_2}{\partial x^2} + \frac{\varepsilon L c_m}{c_q} \frac{\partial \theta_2}{\partial \tau}, \quad x > s(\tau), \quad \tau > 0 \tag{3}$$

$$\frac{\partial \theta_2}{\partial \tau} = a_m \frac{\partial^2 \theta_2}{\partial x^2}, \quad x > s(\tau), \quad \tau > 0. \tag{4}$$

The initial and the boundary conditions are

$$t_1 = t_s \quad \text{at} \quad x = 0, \quad \tau > 0 \tag{5}$$

$$t = t_0 \quad \text{at} \quad \tau = 0, \quad x > 0 \tag{6}$$

$$\theta = \theta_0 \quad \text{at} \quad \tau = 0, \quad x > 0 \tag{7}$$

$$t_1 = t_2 = t_v \quad \text{at} \quad x = s(\tau) \tag{8}$$

$$\theta_1 = \theta_2 = \theta_v \quad \text{at} \quad x = s(\tau) \tag{9}$$

$$k_1 \frac{\partial t_1}{\partial x} - k_2 \frac{\partial t_2}{\partial x} = -(1-\varepsilon)\rho_m L \frac{\mathrm{d}s}{\mathrm{d}\tau} \quad \text{at} \quad x = s(\tau). \tag{10}$$

Symbols are given in the nomenclature. Let

$$\Theta = \frac{\theta-\theta_0}{\theta_v-\theta_0} \tag{11}$$

$$T = \frac{t-t_0}{t_s-t_0} \tag{12}$$

$$\eta = \frac{x}{2\sqrt{(a_q\tau)}} \tag{13}$$

$$T_v = \frac{t_v-t_0}{t_s-t_0} \tag{14}$$

$$Lu = \frac{a_m}{a_q} \tag{15}$$

$$Ko = \frac{Lc_m(\theta_v-\theta_0)}{c_q(t_s-t_0)} \tag{16}$$

$$v = \frac{(1-\varepsilon)\rho_m La_q}{k_1(t_s-t_0)} \tag{17}$$

$$k_{21} = k_2/k_1 \tag{18}$$

and assume

$$s(\tau) = 2\lambda\sqrt{(a_q\tau)} \tag{19}$$

where λ is a constant to be determined later. Assuming T and Θ are functions of η only, equations (1)–(4) are transformed to nondimensional ordinary differential equations.

$$\frac{d^2T_1}{d\eta^2}+2\eta\frac{dT_1}{d\eta} = 0, \quad 0<\eta<\lambda \tag{20}$$

$$\Theta_1 = 1, \quad 0<\eta<\lambda \tag{21}$$

$$\frac{d^2T_2}{d\eta^2}+2\eta\frac{dT_2}{d\eta}-2\varepsilon Ko\eta\frac{d\Theta_2}{d\eta} = 0, \quad \eta>\lambda \tag{22}$$

$$Lu\frac{d^2\Theta_2}{d\eta^2}+2\eta\frac{d\Theta_2}{d\eta} = 0, \quad \eta>\lambda. \tag{23}$$

The boundary conditions (5)–(10) become

$$T_1 = 1 \quad \text{at} \quad \eta = 0 \tag{24}$$

$$T_2 = 0 \quad \text{as} \quad \eta\to\infty \tag{25}$$

$$\Theta_2 = 0 \quad \text{as} \quad \eta\to\infty \tag{26}$$

$$\Theta_1 = \Theta_2 = 1 \quad \text{at} \quad \eta = \lambda \tag{27}$$

$$T_1 = T_2 = T_v \quad \text{at} \quad \eta = \lambda \tag{28}$$

$$\frac{dT_1}{d\eta}-k_{21}\frac{dT_2}{d\eta} = -2v\lambda. \tag{29}$$

Solutions of equations (20) and (23) which satisfy boundary conditions (24), (27), and (28) are easily obtained.

$$T_1 = 1-(1-T_v)\frac{\mathrm{erf}(\eta)}{\mathrm{erf}(\lambda)}, \quad 0<\eta<\lambda \tag{30}$$

$$\Theta_2 = \frac{\mathrm{erfc}(\eta/\sqrt{Lu})}{\mathrm{erfc}(\lambda/\sqrt{Lu})}, \quad \eta>\lambda \tag{31}$$

where

$$\mathrm{erfc}(x) = 1-\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz. \tag{32}$$

Substituting equation (31) into equation (22), and solving the resulting unhomogeneous ordinary differential equation with boundary conditions (25) and (28), one obtains

$$T_2 = \frac{\varepsilon KoLu}{(Lu-1)}\left\{\frac{\mathrm{erfc}(\eta/\sqrt{Lu})}{\mathrm{erfc}(\lambda/\sqrt{Lu})}-\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}\right\} + T_v\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}, \quad \eta>\lambda \tag{33a}$$

when $Lu \neq 1$, and

$$T_2 = \frac{\varepsilon Ko}{\mathrm{erfc}(\lambda)\sqrt{\pi}}\left\{\eta e^{-\eta^2}-\lambda e^{-\lambda^2}\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}\right\} + T_v\frac{\mathrm{erfc}(\eta)}{\mathrm{erfc}(\lambda)}, \quad \eta>\lambda \tag{33b}$$

when $Lu = 1$.

Equations (30), (31), and (33) satisfy all the boundary conditions except (29). Substituting equations (30) and (33) into equation (29), the constant λ is determined from

$$\frac{(1-T_v)}{e^{\lambda^2}\mathrm{erf}(\lambda)}-\frac{k_{21}T_v}{e^{\lambda^2}\mathrm{erfc}(\lambda)}+\frac{\varepsilon Kok_{21}Lu}{(Lu-1)} \times\left\{\frac{1}{e^{\lambda^2}\mathrm{erfc}(\lambda)}-\frac{1}{(\sqrt{Lu})e^{\lambda^2/Lu}\mathrm{erfc}(\lambda/\sqrt{Lu})}\right\} = (\sqrt{\pi})v\lambda \tag{34a}$$

when $Lu \neq 1$, and

$$\frac{(1-T_v)}{e^{\lambda^2}\mathrm{erf}(\lambda)}-\frac{k_{21}T_v}{e^{\lambda^2}\mathrm{erfc}(\lambda)}+\frac{\varepsilon Kok_{21}}{2e^{\lambda^2}\mathrm{erfc}(\lambda)} \times\left\{1-2\lambda^2+\frac{2\lambda}{(\sqrt{\pi})e^{\lambda^2}\mathrm{erfc}(\lambda)}\right\} = (\sqrt{\pi})v\lambda \tag{34b}$$

when $Lu = 1$.

3. DISCUSSION AND CONCLUSIONS

An exact solution to a coupled heat- and mass-transfer problem involving evaporation is obtained. If the constant λ is determined from equation (34), the temperature and moisture distributions are determined from equations (30), (31), and (33), and evaporation thickness is determined from equation (19).

While the approximate solutions of Gupta [4] require to solve three equations for three unknowns, the exact solution obtained in this paper involves only one equation to determine the constant λ.

Functions $e^{\lambda^2}\mathrm{erf}(\lambda)$ and $e^{\lambda^2}\mathrm{erfc}(\lambda)$ which appear in equation (34) are tabulated in [1]. A simple approximate equation to compute $e^{\lambda^2}\mathrm{erfc}(\lambda)$ with errors less than 10^{-8} for $0<\lambda<10$ is also given in [6].

Some results of sample calculations of equation (34) are given in Figs. 1–4. They show that a larger heat of evaporation v gives smaller λ and thus slower evaporation. Figure 2 shows the effect of T_v on evaporation speed; smaller T_v results in faster evaporation. Figure 2 shows faster evaporation when the ratio of thermal conductivity of undried region to that of the dried region k_{21} is higher. Figure 3 indicates that for larger εKo faster drying is obtained for given conditions. Figure 4 shows that larger Luikov number results in faster evaporation. The effect of Luikov number, however, is usually small.

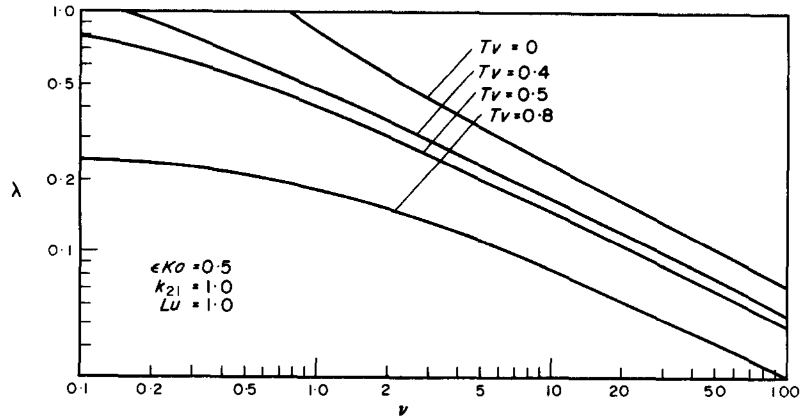

FIG. 1. Solution of equation (34): effect of evaporation temperature.

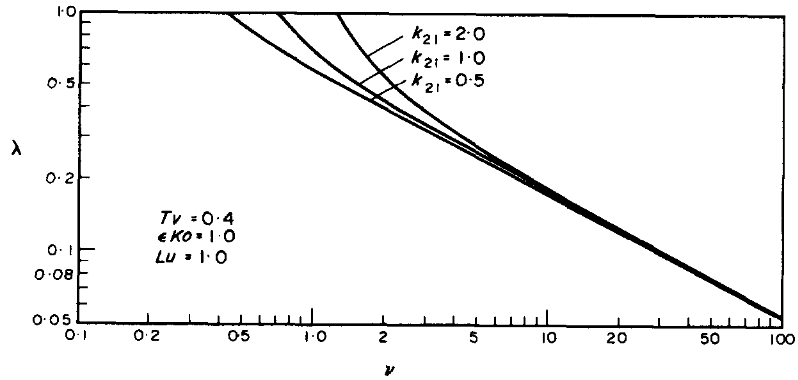

FIG. 2. Solution of equation (34): effect of conductivity ratio.

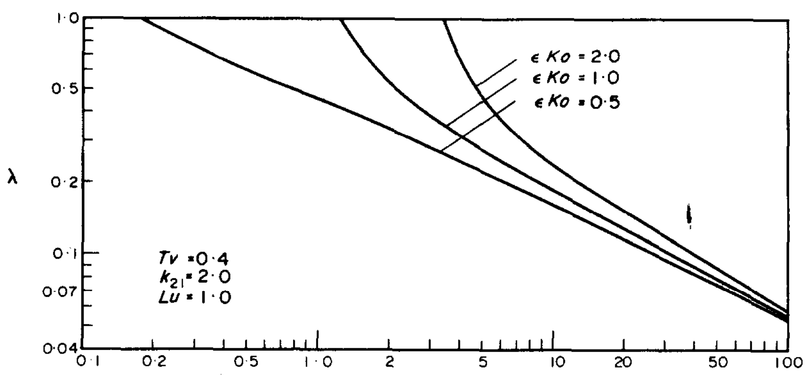

FIG. 3. Solution of equation (34): effect of εKo.

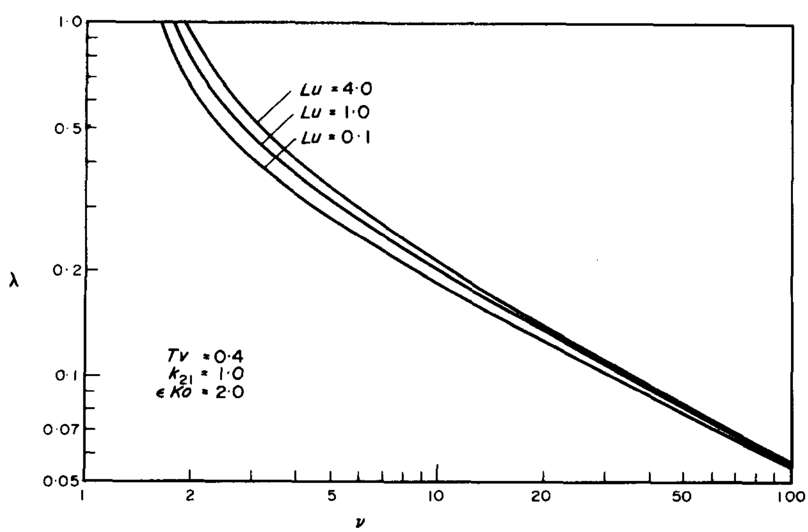

FIG. 4. Solution of equation (34): effect of Luikov number.

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UNE SOLUTION EXACTE DU PROBLEME DE CHANGEMENT DE PHASE COUPLE EN MILIEU POREUX

Résumé—Une solution exacte est donnée au transfert couplé de chaleur et de masse dans un milieu poreux avec évaporation. Les équations aux dérivées partielles fondamentales sont transformées en équations différentielles ordinaires par une transformation de Boltzmann sur les variables. L'épaisseur d'évaporation est proportionnelle à la racine carrée du temps avec une constante de proportionnalité λ obtenue par une équation qui contient uniquement des fonctions tabulées. L'influence des paramètres adimensionnels sur les vitesses d'évaporation est brièvement discutée.

EINE EXAKTE LÖSUNG DES GEKOPPELTEN PHASENÄNDERUNGSPROBLEMS IN EINEM PORÖSEN MEDIUM

Zusammenfassung—Eine exakte Lösung der gekoppelten Wärme- und Stoffübertragung mit Verdampfung in einem porösen Medium wird angegeben. Die gültigen partiellen Differentialgleichungen werden durch Anwendung der Boltzmanntransformation der Variablen in gewöhnliche Differentialgleichungen übergeführt. Die Verdampfungsstärke ist der Quadratwurzel aus der Zeit proportional, wobei der Proportionalitätsfaktor sich aus einer Gleichung ergibt, die nur vertafelte Funktionen enthält. Der Einfluß dimensionsloser Parameter auf die Verdampfungsgeschwindigkeit wird kurz diskutiert.

ТОЧНОЕ РЕШЕНИЕ ЗАДАЧИ СО СЛОЖНЫМ ФАЗОВЫМ ОБМЕНОМ В ПОРИСТОЙ СРЕДЕ

Аннотация — Приводится точное решение задачи сложного тепло- и массообмена в пористой среде при наличии испарения жидкости из пор. С помощью преобразования переменных Больцмана определяющие дифференциальные уравнения в частных производных преобразуются в обычные дифференциальные уравнения. Толщина зоны испарения пропорциональна корню квадратному времени, где коэффициент пропорциональности λ получают из уравнения, содержащего лишь функции, приведенные в таблице. Вкратце рассматривается влияние безразмерных параметров на скорость испарения.